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1993 J. Phys. A: Math. Gen. 26 L69

(http://iopscience.iop.org/0305-4470/26/3/001)

J. Phys. A: Math. Gen. 26 (1993) L69–L73. Printed in the UK

LETTER TO THE EDITOR

A new completeness relation in the q-deformed two-mode Fock space

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Received 1 July 1992

Abstract. In two-mode Fock space a new completeness.relation for the q-analogue of the single-complex-variable-state is proved by virtue of some q-deformed combinatiorial identities, which are also derived here for the first time.

Recently great interest has been paid to the q-analogue of the harmonic oscillators [1–3]. In [4] it is shown that the q-deformed oscillator's algebra can be expressed as

$$aa^+ - q^2 a^+ a = 1 \qquad [N_a, a^+] = a^+ \qquad [N_a, a] = -a \tag{1}$$

where N_a is the number operator defined as

$$[N_a] = a^+ a = (q^{2N} - 1)/(q^2 - 1) \qquad [N_a + 1] = aa^+ \tag{2}$$

or equivalently

$$N_a = \tfrac{1}{2} \ln(1 + (q^2 - 1)a^+ a)/\ln q. \tag{3}$$

The eigenvectors of N_a are given by [4, 5]

$$|n\rangle = a^{+n}/\sqrt{[n]!}\,|0\rangle \qquad [n] = (q^{2n} - 1)/(q^2 - 1) \tag{4}$$

which satisfy

$$a^+|n\rangle = \sqrt{[n+1]}\,|n+1\rangle \qquad a|n\rangle = \sqrt{[n]}\,|n-1\rangle \qquad N_a|n\rangle = n|n\rangle. \tag{5}$$

The resolution of unity for $|n\rangle$ is

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{6}$$

Another completeness relation for the q-analogue coherent state by q-integration is proved in [6]. In the present work we consider the two-mode q-oscillator a^+a and b^+b whose discrete eigenstates $|n, m\rangle$ span a two-mode Fock space. We prove a new completeness relation for the q-analogue of the single-complex-variable-state $|\xi\rangle$. This $|\xi\rangle$, proposed in [7], is defined by the usual Bose operators A^+, B^+ and the usual two-mode vacuum state, its the expression being

$$|\xi\rangle = \exp[-\tfrac{1}{2}|\xi|^2 + \xi A^+ + \xi^* B^+ - A^+ B^+]\,|00\rangle \tag{7}$$

0305-4470/93/030069+05$07.50

which satisfies the following completeness relation by the complex integration

$$\int \frac{d^2\xi}{\pi} |\xi\rangle\langle\xi| = 1 \qquad \xi = |\xi|\, e^{i\theta}. \tag{8}$$

The state $|\xi\rangle$ has some applications in quantum mechanics and quantum optics [8, 9]. The extension of $|\xi\rangle$ to its q-analogue is not trivial because of the q-analogue algebra (1); however, after some effort we introduce the q-analogue state as

$$|\xi\rangle_q = (e_q^{|\xi|^2})^{-1/2} q^{-N_a N_b} (e_q^{a^+ b^+})^{-1} e_q^{\xi a^+} e_q^{\xi^* b^+} |00\rangle \qquad e_q^x = \sum_{n=0}^{\infty} x^n/[n]!. \tag{9}$$

To prove a new completeness relation for $|\xi\rangle_q$ by q-integration over ξ we set up some q-deformed combinatorial identities by introducing a q-analogue binomial theorem. Using these identities we will prove the completeness relation for $|\xi\rangle_q$.

We begin by defining the following q-analogue binomial theorem for real q

$$\prod_{k=0}^{n-1} (x + q^{n-2k-1} y) = \sum_{k=0}^{n} [c_n^k]_q x^k y^{n-k} \tag{10}$$

where

$$[c_n^k]_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{11}$$

which is invariant under the transformation $q \to q^{-1}$ and $[c_{n+m}^n]_q = [c_{n+m}^m]_q$. It is not difficult to derive the q-analogue combinatorial identities from (10)

$$[c_{n+l}^n]_q = \sum_{k=0}^{n} q^{-nl+(n+l)k} [c_n^k]_q [c_l^k]_q \tag{12}$$

$$[c_{n+l}^m]_q = \sum_{k=0}^{m} q^{-ml+(n+l)k} [c_n^{m-k}]_q [c_l^k]_q \tag{13}$$

where n, m, l are non-negative integers and we have defined $[c_m^n]_q = 0$ for $n > m$. Letting

$$[c_m^n] = \frac{[m]!}{[n]!\,[m-n]!} \qquad [n] = q^{n-1}[n]_q \tag{14}$$

we have

$$[c_m^n] = q^{mn-n^2} [c_m^n]_q \qquad [c_{m+n}^n] = [c_{m+n}^m]. \tag{15}$$

Thus we can rewrite (12) and (13) as

$$[c_{n+l}^n] = \sum_{k=0}^{n} q^{2k^2} [c_n^k][c_l^k] \qquad [c_{n+l}^m] = \sum_{k=0}^{m} q^{2k(n-m+k)} [c_n^{m-k}][c_l^k] \tag{16}$$

respectively. Using (16) we can obtain

$$\frac{1}{[n]!\,[l]!} [c_{n+l}^m] = \frac{1}{[n]!\,[l]!} \sum_{k=0}^{m} q^{2k(n-m+k)} [c_n^{m-k}][c_l^k]$$
$$= \sum_{k=0}^{m} \frac{q^{2k(n-m+k)}}{[k]!\,[l-k]!\,[m-k]!\,[n-m+k]!}. \tag{17}$$

From (17) we are led to another identity

$$\sqrt{[c_{n+l}^{n}][c_{n+l}^{m}]}=\sum_{k=0}^{m} q^{2k(n-m+k)}\sqrt{[c_m^k][c_n^{m-k}][c_l^k][c_{n-m+k}^{l-k}]}. \tag{18}$$

We will now prove the completeness relation for (9). Let us first point out that when $q=1$, (9) reduces to $|\xi\rangle$. Surely, (9) is equal to

$$|\xi\rangle_q=(e_q^{|\xi|^2})^{-1/2}q^{-N_aN_b}(e_q^{a^+b^+})^{-1}\sum_{n,m=0}^{\infty}\frac{\xi^n\xi^{*m}}{\sqrt{[n]![m]!}}|n,m\rangle. \tag{19}$$

Note that $|n, m\rangle$ is the usual eigenstate of the number operators N_a and N_b

$$|n,m\rangle=|n\rangle_a\otimes|m\rangle_b.$$

By virtue of the q-combinatorial identities derived above, it is easy to prove some new relations concerning the state $|n, m\rangle$

$$\sum_{l=0}^{\infty}[c_{n+l}^{n}]|l,l\rangle=e_q^{a^+b^+}\sum_{l=0}^{n}q^{2l^2}[c_n^l]|l,l\rangle \tag{20}$$

$$\sum_{l=0}^{\infty}\sqrt{[C_{n+l}^{n}][C_{n+l}^{m}]}|l+n-m,l\rangle$$
$$=e_q^{a^+b^+}\sum_{l=0}^{m}q^{2l(l+n-m)}\sqrt{[c_m^l][c_n^{m-l}]}|l+n-m,l\rangle \tag{21}$$

$$\sum_{l=0}^{\infty}\sqrt{[c_{m+l}^{m}][c_{m+l}^{n}]}|l,l+m-n\rangle$$
$$=e_q^{a^+b^+}\sum_{l=0}^{n}q^{2l(l+m-n)}\sqrt{[c_n^l][c_m^{n-l}]}|l,l+m-n\rangle. \tag{22}$$

With the above mathematical preliminaries we now prove the following completeness relation

$$\int\frac{d_q^2\xi}{\pi}|\xi\rangle_{q\,q}\langle\xi|=1 \tag{23}$$

where the measure of the q-integration is

$$\frac{d_q^2\xi}{\pi}=\frac{dq|\xi|^2\,d\theta}{2\pi}e_q^{|\xi|^2}e_q^{-|\xi|^2}. \tag{24}$$

Substituting (19) into the state $|\xi\rangle_q$ we have

$$\int\frac{d_q^2\xi}{\pi}|\xi\rangle_{q\,q}\langle\xi|=q^{-N_aN_b}(e_q^{a^+b^+})^{-1}\int\frac{d^2q\xi}{\pi}(e_q^{|\xi|^2})^{-1}$$
$$\times\sum_{\substack{n,m=0\\n',m'=0}}^{\infty}\frac{\xi^n\xi^{*m}}{\sqrt{[n]![m]!}}|n,m\rangle\langle n',m'|\frac{\xi^{*n'}\xi^{m'}}{\sqrt{[n']![m']!}}(e_q^{ab})^{-1}q^{-N_aN_b}. \tag{25}$$

Integrating over the angular variable θ from 0 to 2π and using the q-analogue Euler formula [6]

$$\int_0^{\zeta}d_qx\,e_q^{-x}x^n=[n]! \tag{26}$$

where $-\zeta$ is the largest zero of e_q^x, we can rewrite the right-hand side of (25) as

$$q^{-N_aN_b}(e_q^{a^+b^+})^{-1}\sum_{n,m,k=0}^{\infty}\frac{[n+k]!}{\sqrt{[n]![m]![k]![n+k-m]!}}|n,m\rangle\langle n+k-m,k|(e_q^{ab})^{-1}q^{-N_aN_b}. \tag{27}$$

After some rearranging of the terms in the iterated series we may express (27) as

$$q^{-N_aN_b}(e_q^{a^+b^+})^{-1}\sum_{l=0}^{\infty}\sum_{n,m=0}^{l}\sqrt{[c_l^n][c_l^m]}\,|n,m\rangle\langle l-m,l-n|(e_q^{ab})^{-1}q^{-N_aN_b}. \tag{28}$$

On the other hand, we know

$$1=\sum^{\infty}\,|n,m\rangle\langle n,m|=\sum^{\infty}\sum^{l}\,|k,l-k\rangle\langle k,l-k|.$$

It then turns out that proving the completeness relation (23) is equivalent to verifying the following relation:

$$\sum_{l=0}^{\infty}\sum_{n,m=0}^{l}\sqrt{[c_l^n][c_l^m]}\,|n,m\rangle\langle l-m,l-n|(e_q^{ab})^{-1}q^{-N_aN_b}$$
$$=e_q^{a^+b^+}q^{N_aN_b}\sum_{l=0}^{\infty}\sum_{k=0}^{l}|k,l-k\rangle\langle k,l-k|. \tag{30}$$

Examining the right-hand side of (30) we find

RHS of (30)

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{l}\sum_{n=0}^{\infty}q^{k(l-k)}\sqrt{[c_{n+k}^k][c_{l+n-k}^{l-k}]}\,|k+n,l+n-k\rangle\langle k,l-k|$$
$$=\sum_{L=0}^{\infty}\sum_{N=0}^{L}\sum_{M=L-N}^{L}q^{(L-M)(L-N)}\sqrt{[c_N^{L-M}][c_M^{L-N}]}\,|N,M\rangle\langle L-M,L-N|. \tag{31}$$

The left-hand side of (30) can then be separated into three parts according to $n=m$, $n>m$ and $n<m$

LHS of (30)

$$=\left\{\sum_{n=0}^{\infty}|n,n\rangle\sum_{l=0}^{\infty}[c_{l+n}^n]\langle l,l|+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}|n,m\rangle\right.$$
$$\times\sum_{l=0}^{\infty}\sqrt{[c_{n+l}^n][c_{n+l}^m]}\,\langle l+n-m,l|$$
$$\left.+\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}|n,m\rangle\sum_{l=0}^{\infty}\sqrt{[c_{m+l}^m][c_{m+l}^n]}\,\langle l,l+m-n|\right\}(e_q^{ab})^{-1}q^{-N_aN_b}. \tag{32}$$

Substituting equations (20), (21) and (22) into the above expression we obtain

$$(32)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}q^{l^2}[c_l^n]|n,n\rangle\langle l,l|$$
$$+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{l=0}^{m}q^{l(l+n-m)}\sqrt{[c_m^l][c_n^{m-l}]}\,|n,m\rangle\langle l+n-m,l|$$
$$+\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\sum_{l=0}^{n}q^{l(l+m-n)}\sqrt{[c_n^l][c_m^{n-l}]}\,|n,m\rangle\langle l,l+m-n| \tag{33}$$

where we have used

$$q^{N_a N_b}|n, m\rangle = q^{nm}|n, m\rangle. \tag{34}$$

In order to compare (31) with (33), we notice that (31) considered as a sum concerning the ket vectors $|N, M\rangle$ where both N and M run over all non-negative integers, so we may also separate the sum in (31) into three parts $N=M$, $N>M$ and $N<M$. For the $N=M$ part from (31) we get

$$\sum_{N=0}^{\infty}\sum_{L=N}^{2N} q^{(L-N)^2}[c_N^{L-N}]|N, N\rangle\langle L-N, L-N| = \sum_{N=0}^{\infty}\sum_{L=0}^{N} q^{L^2}[C_N^L]|N, N\rangle\langle L, L|$$

which is exactly the same as the first term of (33). For the $N>M$ part from (31) we get

$$\sum_{N=1}^{\infty}\sum_{M=0}^{N-1}\sum_{L=N}^{N+M} q^{(L-M)(L-N)}\sqrt{[c_N^{L-M}][c_M^{L-N}]}\,|N, M\rangle\langle L-M, L-N|$$
$$= \sum_{N=1}^{\infty}\sum_{M=0}^{N-1}\sum_{L=0}^{M} q^{L(L+N-M)}\sqrt{[c_M^L][c_N^{M-L}]}\,|N, M\rangle\langle L+N-M, L|$$

which coincides with the second term of (33). Finally for the $N<M$ part from (31) we obtain

$$\sum_{M=1}^{\infty}\sum_{N=0}^{M-1}\sum_{L=M}^{N+M} q^{(L-M)(L-N)}\sqrt{[c_N^{L-M}][c_M^{L-N}]}\,|N, M\rangle\langle L-M, L-N|$$
$$= \sum_{M=1}^{\infty}\sum_{N=0}^{M-1}\sum_{L=0}^{N} q^{L(L+M-N)}\sqrt{[c_N^L][c_M^{N-L}]}\,|N, M\rangle\langle L, L+M-N|$$

which is equal to the third term of (33). Summarizing the above analysis we reach the conclusion that (31) is exactly the same as (33), which means that the completeness relation (23) for $|\xi\rangle_q$ is proved.

This work is supported by the PhD directing programme foundation of Chinese Universities.

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